Investigating Finite Difference Methods for Option Pricing

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Abstract

We investigate finite difference methods for option pricing, focusing mainly on digital options. *We also study how the application of those techniques performs in terms of quality for* vanilla options. *The methods considered are the basic explicit finite difference scheme and the Crank-Nicolson scheme. Furthermore we investigate aspects of coordinates transformation to solve PDEs over non-uniform grids.*

1 Introduction

The main goal of finite difference techniques is solve numerically the *Black-Scholes* equation or one if its variations. The aim of build a numerical scheme for that equation is not to find the solution itself (we know that Black-Scholes for European options has an analytical solution) but to exploit such scheme to solve more general equations and inequalities. This means to be able to price more exotic option: in our case, *digital options*. An easy way to start is to impose a coordinate transformation that permits to simplify the BS equation to one of its variances with constant coefficients.

2 From Black-Scholes to a finite difference scheme

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

With *V(S,t)* the option price, *S* the stock price, *r* the interest rate, σ the volatility term and T the maturity. It is easy to prove that the process underlying the BS equation is a *diffusion* process. In fact, imposing:

- $x = ln(S/K) + (r - \frac{\sigma^2}{2})(T - t)$
- $\tau = T - t$
- $u = Ve^{r(T-t)}$

we obtain the following equation:

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$$

which corresponds exactly to a time dependent diffusion process with diffusion coefficient $D = \frac{\sigma^2}{2}$. However for our studies we'll concentrate on the following constant coefficients transformation:

- solve backward in time: $\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$
- logarithmic scale over S: $x = ln(S)$

that leads BS to:

$$\frac{\partial V}{\partial \tau} = (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - rV$$

with the corresponding explicit finite difference schemes that reads:

$$\begin{aligned} V_i^{n+1} &= V_i^n + (r - \frac{\sigma^2}{2})\frac{\delta\tau}{2\delta x}(V_{i+1}^n - V_{i-1}^n) \\ &+ \frac{\sigma^2}{2}\frac{\delta\tau}{\delta x^2}(V_{i+1}^n - 2V_i^n + V_{i-1}^n) - r\delta\tau V_i^n \end{aligned}$$

We now entirely dropped the *S* term, furthermore the logarithmic transformation of *S* implies a finer grid resolution at the beginning of the scale, and a coarser grid resolution for high stock prices. This can be an advantage if we are more interested in the behavior of the option price for small values of *S*.

2.1 Boundary and initial conditions

The first problem raised by the discussed finite difference scheme is defining the boundary conditions. If

$$S \to 0$$

then

$$x \to -\infty$$

nevertheless a value of $S = 0$ doesn't make sense in the stock market, where the stock would stop to exist. We therefore have to consider a value $\epsilon > 0$ as S_{min} with

$$V(S_{min}, \tau) = 0$$

From the other side we know that for:

$$S \to \infty$$

then

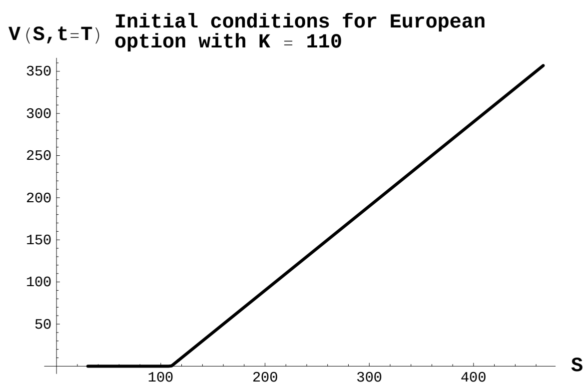
$$x \to \infty$$

We impose an S_{max} as big as necessary with a discounted option price of

$$V(S_{max}, \tau) = Max(S_{max} - K, 0)e^{-r(T-\tau)}$$

For our studies we limited the stock prices between $S_{min} = 0.25 \cdot K$ and $S_{max} = 4 \cdot K$.

Regarding initial conditions the problem is simple: we are solving backward in time, thus the initial values are the values at time of maturity, that are known from the definition of the option. An example of initial conditions for an European option is:

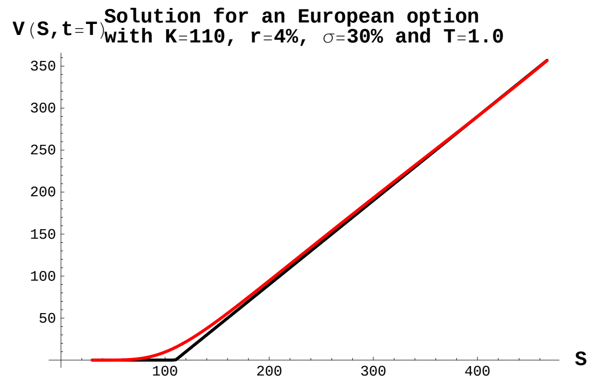


3 Pricing European options

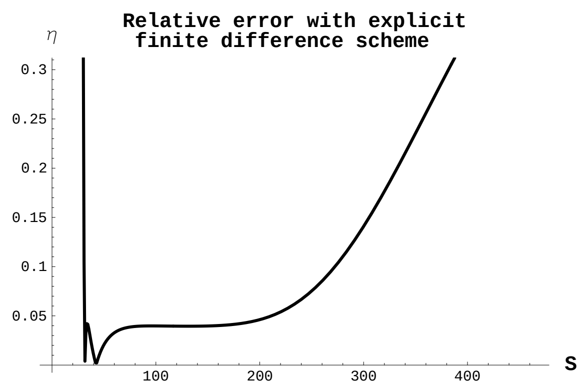
We now price the following European option with the finite difference scheme discussed previously:

- $K = 110$
- $r = 4\%$
- $\sigma = 30\%$
- $T = 1$
- $\delta x = 0.0277$
- $\delta\tau = 0.002$

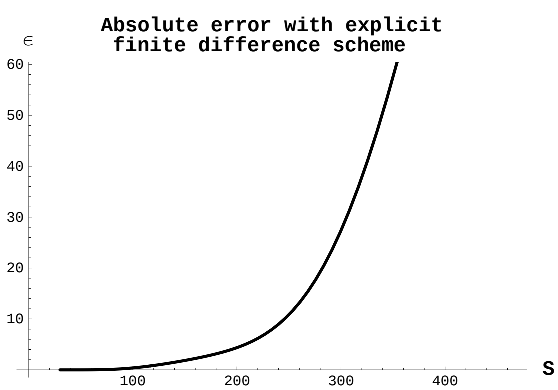
The solution is:



With a the following relative error (compared with the Black-Scholes' analitycal solution):



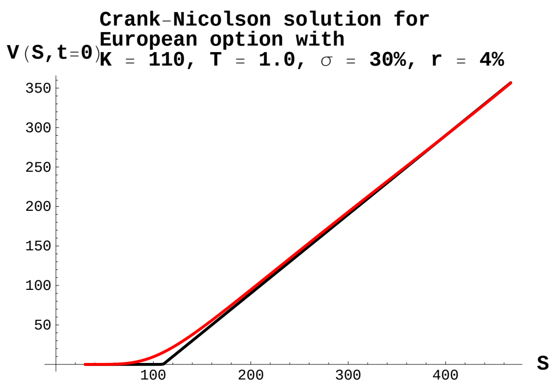
With the absolute error with respect to the analytical solution of Black-Scholes:



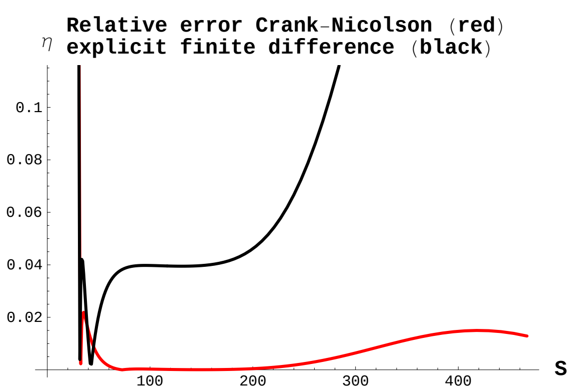
The results agree with what we expected: this finite difference method in fact has an accuracy of $O(\delta x^2, \delta\tau)$ and inside the S space δS grows exponentially. This behavior of the error has to be taken into the account also examining the results presented in the next sections. Moreover observing the relative error some precision error raise close to S_{min} as well.

3.1 A further step: Crank-Nicolson

A more accurate method to solve numerically a PDE (Partial Differential Equation) is the *Crank-Nicolson* algorithm, discussed in the next sections. In this case the accuracy is higher: $O(\delta x^2, \delta t^2)$ and the stability is now unconditioned. The result for the same option using the Crank-Nicolson scheme is:



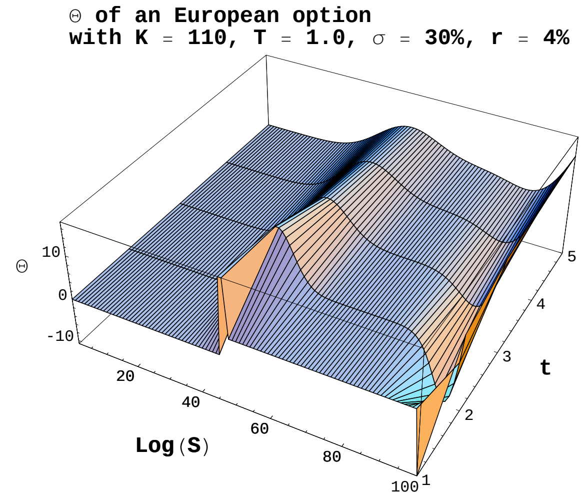
while the error, compared to the finite difference is:



as we expected, this method performs better due to his higher accuracy on time.

3.2 Greeks

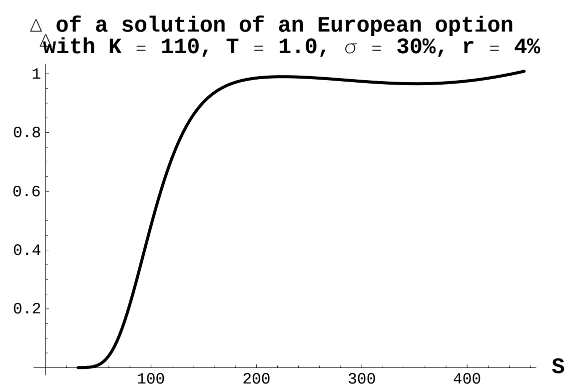
Computing the greeks with finite difference is quite straight forward. The derivatives are computed numerically over the entire 3-dimensional space (S, t) for Θ, over the solution for Δ and Γ and over the solution at strike price for ρ and $Vega$.



3.2.1 $\Theta = \frac{\partial V}{\partial t}$

Θ shows clearly the underlying diffusion process: the main variation in time is located close to the strike price K and close to the left boundary due to the approximation we made in the S domain.

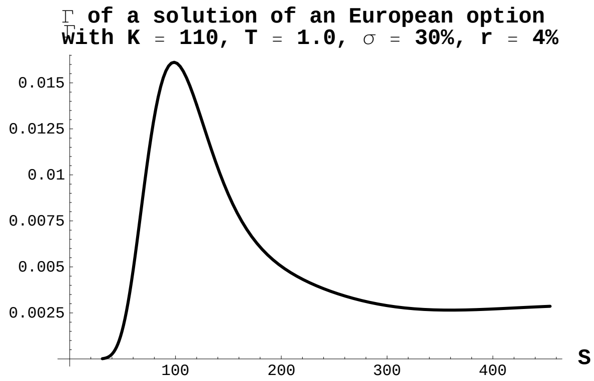
3.2.2 $\Delta = \frac{\partial V}{\partial S}$



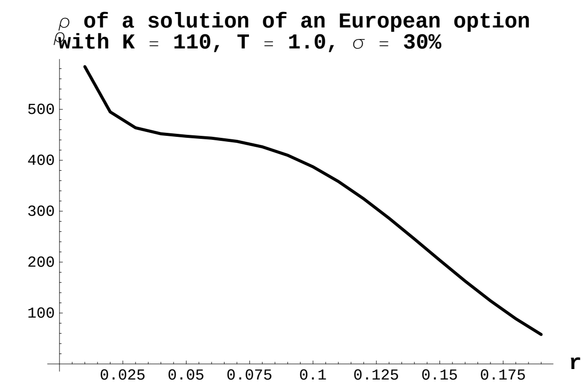
Δ shows how the option price changes with respect to the stock price. With $r = \sigma = 0$ this greek would draw a step function while for values > 0 the step function becomes smoother while tending to 1. The Δ is a crucial value for risk edging cause it defines the ratio between long a short position in a *risk-less* portfolio.

3.2.3 $\Gamma = \frac{\partial V}{\partial^2 S}$

The variation of the Δ with respect to the stock price is represented as Γ. As we expected the largest changes of Δ are located around the strike price.

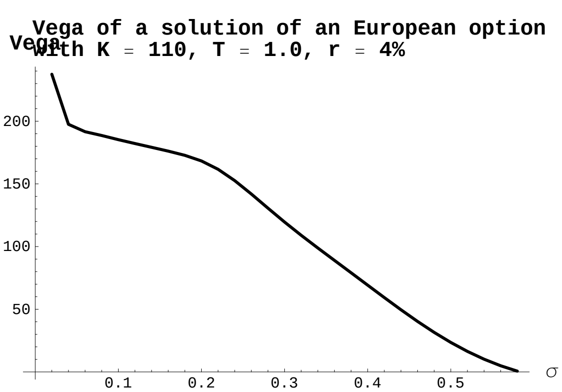


3.2.4 $\rho = \frac{\partial V}{\partial r}$



The sensitivity of the payoff with respect to the interest rate *r* tends to decrease when raising *r* until it reachs a *saturation* point.

3.2.5 $Vega = \frac{\partial V}{\partial \sigma}$



Vega shows that the derivative of the option price over σ tends to decrease for high values of σ. It actually behaves similarly to ρ, probably the reason is because both σ and r participate on the diffusion coefficient of the underlying process. High values of those parameters increase the diffusion speed that, however, reachs its natural stability described for generic two-dimensional harmonic fields by the *Laplace* equation:

$$\nabla^2 \Phi = 0$$

4 Stability and accuracy analysis

The results obtained with both methods we used are subjected to errors in several ways. In general the existence of errors must be accepted, but not their magnitude.

- Modeling error

  The equations of the model are only approximations of the reality.

- discretization errors

  By discretizing the PDE we introduce the so called discretization errors. This happens when the PDE is replaced by a set of approximating polynomial equations from its *Taylor* expansion. An essential portion of the discretization errors is the error between differential quotients and difference quotients. Another smaller influence error is the truncating of the infinite interval to a finite interval, the implementation of the boundary conditions.

- error from solving the linear equation

  An iterative solution of the linear systems of equation $A\omega = b$ means that the error approaches 0 when $k \to \infty$, where k counts the number of iterations. In general one has no accurate information on the size of these errors. Typically the modelling errors are larger than the discretization errors. Here we analyze the discretization error when solving the diffusion equation.

4.1 Crank-Nicolson

In the mathematical subfield numerical analysis, the Crank-Nicolson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. It is a second-order method in space, implicit in time, and numerically stable. We consider the equivalent diffusion equation of Black-Scholes:

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

Crank and Nicolson suggested to average the forward and the backward difference method. Forward for v we have:

$$\frac{\omega_{i,v+1}-\omega_{iv}}{\Delta\tau}=\frac{\omega_{i+1,v}-2\omega_{iv}+\omega_{i-1,v}}{\Delta x^2}$$

Backward for $v+1$ we have:

$$\frac{\omega_{i,v+1}-\omega_{iv}}{\Delta\tau}=\frac{\omega_{i+1,v+1}-2\omega_{i,v+1}+\omega_{i-1,v+1}}{\Delta x^2}$$

Addition of these two yields:

$$\frac{\omega_{i,v+1}-\omega_{iv}}{\Delta\tau}=\frac{\omega_{i+1,v+1}-2\omega_{i,v+1}+\omega_{i-1,v+1}}{\Delta x^2}\dots$$

The equation above involves in each of the time levels v and $v+1$ three values of ω. This is the basis for an efficient method. Its features are as follows:

Theorem (*Crank-Nicolson*)
Suppose y is smooth in the sense $y\in C^4$. Then:

- The order of the method is $O(\Delta\tau^2)+O(\Delta x^2)$.

  Proof:
  Another notation from the symmetric difference quotient of second order for y_{xx} is

  $$\delta_x^2\omega_{iv}:=\frac{\omega_{i+1,v}-2\omega_{iv}+\omega_{i-1,v}}{\Delta x^2}$$

  First apply the operator δ_x^2 to the exact solution y. Then by Taylor expansion for $y\in C^4$ it can be shown that

  $$\delta_x^2 y_{iv}=\frac{\partial^2}{\partial x^2}y_{iv}+\frac{\Delta x^2}{12}\frac{\partial^4}{\partial x^4}y_{iv}+O(\Delta x^4)$$

  The local discretization error is:

  $$\epsilon=O(\Delta\tau^2)+O(\Delta x^2)$$

- For each v a linear system of a simple tridiagonal structure must be solved.

  With $\lambda:=\frac{\Delta\tau}{\Delta x^2}$ the equation 4.12 is rewritten

  $$-\frac{\lambda}{2}\omega_{i-1,v+1}+(1+\lambda)\omega_{i,v+1}-\frac{\lambda}{2}\omega_{i+1,v+1}=\frac{\lambda}{2}\omega_{i-1,v}+(1-\lambda)\omega_{iv}+\frac{\lambda}{2}\omega_{i+1,v}$$

  For the simplest boundary conditions equation 4.14 is a system of equations. With matrices

  $$A:=\begin{pmatrix}1+\lambda & -\frac{\lambda}{2} & & 0\\ -\frac{\lambda}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots\\ 0 & & \ddots & \ddots\end{pmatrix}\dots$$

  $$B:=\begin{pmatrix}1-\lambda & \frac{\lambda}{2} & & 0\\ \frac{\lambda}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots\\ 0 & & \ddots & \ddots\end{pmatrix}$$

  the system above can be rewritten as

  $$A\omega^{(v+1)}=B\omega^{(v)}$$

  The eigenvalues of A are real and lie between 1 and $1+2\lambda$. It follows that A is nonsingular and the solution of (4.15b) is uniquely defined.

- Stability holds for all $\Delta\tau\geq 0$. The matrices A and B can be written in terms of a constant tridiagonal matrix,

  $$A=I+\frac{\lambda}{2}G,$$

  $$G:=\begin{pmatrix}2 & -1 & & 0\\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots\\ 0 & & \ddots & \ddots\end{pmatrix}$$

  $$B=I-\frac{\lambda}{2}G$$

  Now the equation (4.15b) reads

  $$\begin{aligned}(2I+\lambda G)\omega^{(v+1)}&=(2I-\lambda G)\omega^{(v)}\\&=(4I-2I-\lambda G)\omega^{(v)}\\&=(4I-C)\omega^{(v)}\end{aligned}$$

which leads to the formally explicit iteration

$$\omega^{(v+1)}=(4C^{-1}-I)\omega^{(v)}$$

If we denote the eigenvalues of C with μ_k^C, then

$$\mu_k^C = 2 + 4\lambda \sin^2 \frac{k\pi}{2m}$$

We require for a stable method that for all k,

$$\left| \frac{4}{\mu_k^C} - 1 \right| < 1$$

This is guaranteed because all the eigenvalues of C are bigger than 2. As a result, Crank-Nicolson method is unconditionally stable for all $\lambda > 0 (\Delta\tau > 0)$. As shown analytically, the Crank-Nicolson method is accurate up to $O(\delta\tau^2, \delta x^2)$ while the explicit method is accurate up to $O(\delta\tau, \delta x^2)$.

This means that increasing the number of time-steps (making $\delta\tau$ smaller) while keeping the τ_{max} fixed will result in smaller errors when using Crank-Nicolson than using explicit method.

However, since the error terms contain both terms δx^2 and $\delta\tau^2$, only making the time-step smaller and smaller while keeping δx fixed will eventually make $\delta\tau$ smaller than δx^2. If then $\delta\tau << \delta x^2$, making $\delta\tau$ smaller will not make the overall errors smaller since δx^2 will dominate the error terms of order $\delta\tau$ and higher.

Especially, if we make the mistake of keeping α fixed as we make the time-step smaller, the Crank-Nicolson does not converge faster than the explicit method, since for each fixed α, $\delta\tau = \alpha\delta x^2$ so even the Crank-Nicolson will converge like

$$O(\delta\tau^2, \delta x^2) = O(\delta x^4, \delta x^2) = O(\delta x^2)$$

when

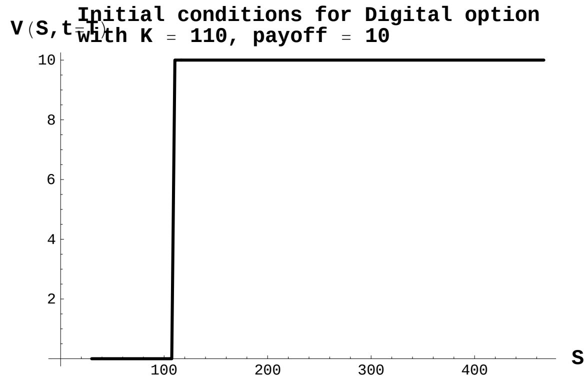
$$\delta x << 1$$

Below we prove this by testing our implementation with different values of δx and $\delta\tau$.

5 Digital options

A digital option is finance derivative that pays either a fixed amount C if at maturity the stock price is bigger than the strike price or zero otherwise. The main characteristics of a digital option thus are:

- the payoff doesn't depend on $S_i - K$
- if $S > K$ payoff = Const, otherwise payoff = 0
- the initial condition is a step function with the step on K
- boundary conditions are trivial (0 and C)

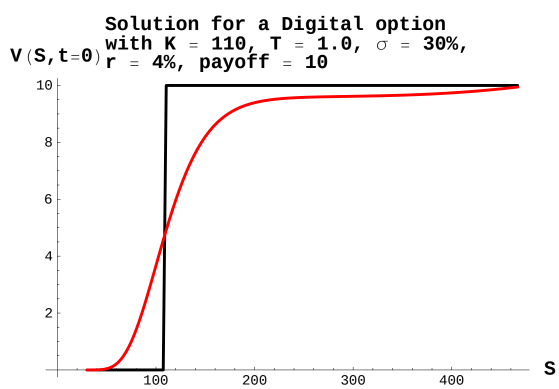


- in many cases the result curve is not smooth: we need a different coordinates transformation

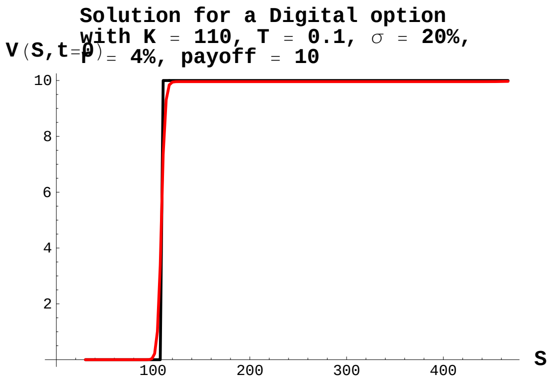
First we use the implemented Crank-Nicolson implicit scheme to compute a digital option with the following parameters:

- $K = 100$
- $\sigma = 30\%$
- $r = 4\%$
- $\delta x = 0.0277$
- $\delta\tau = 0.002$

The result is:

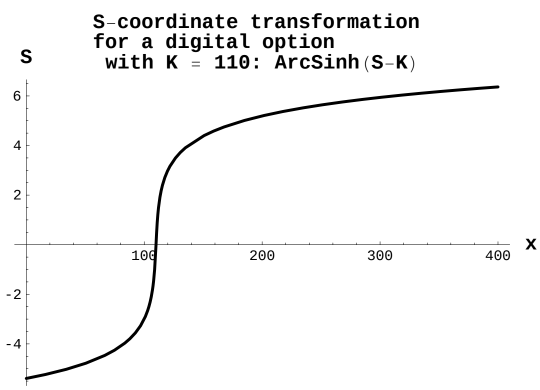


however for smaller values of σ or small maturities the precision of the curve could not be enough satisfactory:



In order to obtain a finest accuracy around the interesting point (*K*) we propose to apply the following transformation instead of *ln(S)*:

$$\begin{aligned} x &= ArcSinh(S-K) \\ S &= Sinh(x)+K \end{aligned}$$



The *ArcSinh(x)* maps the *S* space into the *x* space producing a finest grid around the strike price *K* and a coarser grid far from that value. In this way we can reduce the error near to the interesting points while neglecting larger errors far from the strike price. Here we expose precisely how we derived a finite difference scheme with this coordinates transformation.

5.1 Derivation

Black-Scholes:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial t^2}+rS\frac{\partial V}{\partial S}-rV=0$$

We impose the transformation:

$$\begin{aligned} x &= ArcSinh(S-K) \\ S &= Sinh(x)+K \\ \frac{\partial V}{\partial t} &= -\frac{\partial V}{\partial \tau} \end{aligned}$$

For the chain rule we have:

$$\frac{\partial V}{\partial S}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial S}=\frac{\partial V}{\partial x}\frac{1}{\sqrt{1+(S-K)^2}}=\frac{\partial V}{\partial x}\frac{1}{\sqrt{1+Sinh^2(x)}}$$

And:

$$\begin{aligned} \frac{\partial^2 V}{\partial S^2} &= \frac{\partial^2 V}{\partial x^2}(\frac{\partial x}{\partial S})^2+\frac{\partial V}{\partial x}\frac{\partial^2 x}{\partial S^2} \\ &= \frac{\partial^2 V}{\partial x^2}\frac{1}{1+Sinh^2(x)}+\frac{\partial V}{\partial x}(-\frac{Sinh(x)}{(1+Sinh^2(x))^{3/2}}) \end{aligned}$$

The Black-Scholes equation becomes then:

$$\begin{aligned} -\frac{\partial V}{\partial \tau} &+ \frac{1}{2}\sigma^2(Sinh(x)+K)^2(\frac{\partial^2 V}{\partial x^2}\frac{1}{1+Sinh^2(x)} \\ &+ \frac{\partial V}{\partial x}(-\frac{Sinh(x)}{(1+Sinh^2(x))^{3/2}})) \\ &+ r(Sinh(x)+K)\frac{\partial V}{\partial x}\frac{1}{\sqrt{1+Sinh^2(x)}}=0 \end{aligned}$$

Simplifying:

$$-\frac{\partial V}{\partial \tau}+\alpha\frac{\partial V}{\partial x}+\beta\frac{\partial^2 V}{\partial x^2}-rV=0$$

With:

$$\begin{aligned} \alpha &= \frac{r(Sinh(x)+K)}{\sqrt{1+Sinh^2(x)}}-\frac{\sigma^2(Sinh(x)+K)^2 Sinh(x)}{2(1+Sinh^2(x))^{3/2}} \\ \beta &= \frac{\sigma^2(Sinh(x)+K)^2}{2(1+Sinh^2(x))} \end{aligned}$$

Using Taylor:

$$\begin{aligned} - & \frac{V_i^{t+1}-V_i^t}{\delta\tau}+\alpha\frac{V_{i+1}^t-V_{i-1}^t}{2\delta x} \\ + & \beta\frac{V_{i-1}^t-2V_i^t+V_{i+1}^t}{\delta x^2}-rV_i^t=0 \end{aligned}$$

Finally, the corresponding explicit finite difference scheme for this transformation reads:

$$\begin{aligned} V_i^{t+1} &= V_i^t+\alpha\frac{\delta\tau}{2\delta x}(V_{i+1}^t-V_{i-1}^t) \\ &+ \beta\frac{\delta\tau}{\delta x^2}(V_{i-1}^t-2V_i^t+V_{i+1}^t)-r\delta\tau V_i^t \end{aligned}$$

Unfortunately, besides we believe our scheme is correct, we couldn't make the algorithm work properly. We still have some stability problems that we are investigating. Nevertheless we really think this transformation could strongly improve the accuracy of Digital options pricing where is needed. It is moreover easy to port this technique to other problems where is necessary to solve PDEs in specific interesting points.

6 Conclusions

We investigated how to derive finite difference schemes from Black-Scholes and how to implement them as explicit or implicit form (with Crank-Nicolson). The latter one in particular, represents one of the main implicit schemes for PDE solving due to its higher accuracy. One of the main advantages of finite difference methods is the easiness on finding the *Greeks* even for some tricky options like Digital options. An interesting part of our study has concentrated on coordinate transformations in order to solve finance PDEs over non-uniform grids. This aspect, depending on the kind of financial derivative we're dealing with, is crucial in terms of error minimization. In many problems indeed we are interested in few particular points, most of the times even *one* point where we want to concentrate the highest accuracy of a finite difference scheme. Finally, performing a deep analytical and numerical study is absolutely fundamental in financial problems. Many problems related to stability and accuracy are often encountered, moreover the problem needs to be analyze considering at the same time both its mathematical and numerical background and the real financial world where it exists.